Vision - Potential

Vision Within Every Instructor - Potential Within Every Student

Newsletter of the HBCU College Algebra Reform Consortium*
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[1] Geometry - Algebra Interplay

Welcome back to school and, in particular, to the wonderful, challenging, and useful world of Contemporary College Algebra with its emphasis on problem-solving. A common ingredient in everyone's problem-solving process should be **Sketch a Picture**. This is where geometry enters the picture (no pun intended). Lines, rectangles, circles, parabolas, etc. are common geometric shapes that we have known for years. Linking these geometric shapes with their algebraic equations (e.g., the equation of circle with center (a,b) and radius r is $(x-a)^2+(y-b)^2=r^2$) forms "bridges" that allow us to move back and forth between geometry and algebra, each enhancing the other. For example solving (or approximating the solutions) $2^x = x^2$ for x is an algebraic problem. However it is best accomplished by plotting $f(x) = 2^x - x^2$ and noting where the curve touches the x-axis, a geometric problem. Another example is determining if two lines are parallel, a geometric problem. This question is best resolved by writing the equations of the lines in point slope form to see if their slopes are equal, an algebraic problem.

* Supported by the U.S. Military Academy, West Point, NY.

Triangles, particularly right-triangles, are another basic shape along with its algebraic counterpart, the Pythagorean theorem, that is frequently encountered. (Pythagorean theorem relates the sides and hypotenuse of a right triangle: *The sum of the squares of the sides equals the square of the hypotenuse.*) Consider their critical role in modeling periodic motion. Periodicity is central to our lives - the days of the week, the seasons, hours of daylight, heartbeats, ocean tides, etc. are a few examples. Maybe the most familiar example, is the periodic motion of the minute hand on a clock (non-digital). The tip of the minute hand traces out a complete circle every hour. Visualize the circle on the xy-plane with the center at the origin and then visualize a right triangle formed by dropping a line from the tip of the minute hand to the x-axis and a line (hypotenuse) from the tip of the minute hand to the origin. As the tip of the hour hand travels around the circle both the height and the base of the right-triangle change in a periodic manner. Trigonometric functions are defined to describe this motion ($\sin(\theta) = \frac{\text{side opposite}}{\text{hypotenuse}}, \cos(\theta) = \frac{\text{side adjacent}}{\text{hypotenuse}}$). Using these functions, the Pythagorean theorem becomes the fundamental trigonometric identity: $\sin^2(\theta) + \cos^2(\theta) = 1$.

Draftspeople, engineers, kitchen planners, and a host of people in other professions often encounter a need to shift a shape (circle, parabola, etc.) from one position to another or to resize or to reorient it in their drawings. (See Section 3.4 in *Contemporary College Algebra: Data, Functions, Modeling* on "Shifting and Scaling Graphs".) The following four articles illustrate the interplay between geometry and algebra in resolving standard problems in drafting.

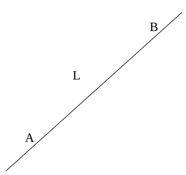
[2] Finding a Second Point

Determining the distance between two points is a straightforward calculation using the distance formula. However, determining the coordinates of a point on a line that is a given distance from a known point on the line presents an interesting challenge. Consider the line defined by $y = 2x + 5$ and the point A: (1,7). Determine a point B on the line that is 3 units from point A. (How many solutions should there be?) Hint: Denote the coordinates of B by (a,b). What relationship must exist between a and b for B to lie on the line? Set the distance between A and B equal to 3 and solve for a and b.

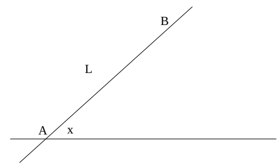
[3] Slopes of Perpendicular lines

Are the lines $y = 3x + 5$ and $y = -2x + 6$ perpendicular to each other? How can you tell? The following outline of a geometric argument provides an answer.

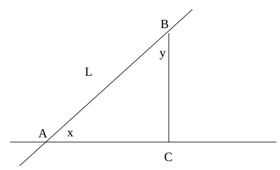
1. Consider a line L with distinct points marked A and B on it.



2. Draw a horizontal line through point A and denote the angle formed by x.

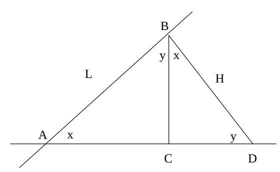


3. Draw a line through B perpendicular to the horizontal line forming a right triangle ACB.



   Explain why angle $y = \frac{\pi}{2} - x$.

4. Draw a line, call it H, through B perpendicular to L. Let D denote the point of intersection of H with the horizontal line.



   Explain the labelling of the angles in the right triangle BCD.

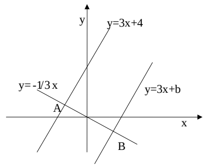
5. The slope of L is $\tan(x) = \frac{BC}{AC}$ and the slope of line H is $\tan(\pi - y) = -\tan(y) = -\frac{AC}{BC}$. Therefore the slope of H $= -\frac{1}{\text{slope of L}}$.

   Thus the slopes of perpendicular lines are negative reciprocals of each other. (The reader should verify that the result holds if points A and B are interchanged or if line L has a negative slope.)

[4] Parallel Lines

A draftsman needs to determine the equation of a line parallel to the line L: $y = 3x + 4$ and 3 units away from L. He knows that parallel lines have the same slope. Therefore line

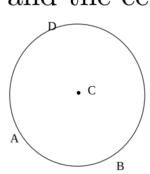
$y = 3x + b$ is parallel to L for any value of b. The problem is how to determine b so that the two lines will be 3 units apart. The draftsman also knows (having read the previous article) that any line with slope $-1/3$ is perpendicular to L. Thus line $y = -\frac{1}{3}x$ is perpendicular to L and to the arbitrary line $y = 3x + b$. The question can now be rephrased to: Determine the value of b so that the intersection points of $y = -\frac{1}{3}x$ with line L and the arbitrary line $y = 3x + b$ are 3 units apart. A picture helps to show the reasoning.



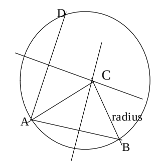
The coordinates of point A are found by solving the system $\begin{cases} y = -\frac{1}{3}x \\ y = 3x + 4 \end{cases}$ and the coordinates of point B are found by solving the system $\begin{cases} y = -\frac{1}{3}x \\ y = 3x + b \end{cases}$. Note these coordinates are in terms of the unknown b. The value of b can now be determined by setting the distance between points A and B equal to 3 and solving for b.

[5] Three Points Determine a Circle

Problem: Determine the equation of the circle whose circumference contains 3 points: A, B, and D. Thus we need to determine the radius and the center, C.



We may begin by investigating a possible relation between the center of the circle, C, and a chord, say AB. Lines drawn from C to A and from C to B are radii of the circle and thus are equal. Therefore triangle ACB is an isosceles triangle (triangle with 2 equal sides). The bisector of the angle ACB is also the perpendicular bisector of the chord AB. (Show why this is true.) Hence the perpendicular bisectors of chords AB and AD intersect at C, the center of the circle.



Knowing how to determine the midpoint of a line segment and that the slopes of perpendicular lines are negative reciprocals, allows us to determine the two perpendicular bisectors. The coordinates for the center of the circle are then found by solving the system consisting of these two perpendicular bisectors. Knowing the coordinates of the center, it is easy to find the radius and then write the equation for the circle. The reader should pick three non-colinear points and then determine the equation of the circle they determine.

As another approach and as a check on your computations, substitute the coordinates of each of the three points into the general equation of a circle centered at (a,b) with radius r, $(x - a)^2 + (y - b)^2 = r^2$. Now solve the resulting system of three equations for a, b, and r.

[6] Arithmetic and Geometric Means

The arithmetic mean of two numbers, x and y, is $\frac{x+y}{2}$ and the geometric mean is $\sqrt{xy}$. Discuss how the two means compare with respect to size? That is, is the arithmetic mean always greater than or equal to the geometric mean or is the geometric mean always greater than or equal to the arithmetic mean or are there pairs of numbers for which the arithmetic mean is greater than the geometric mean and other pairs for which the reverse inequality holds? Also are there pairs of numbers for which both means are equal? Explain your reasoning.

As a follow-on problem, define arithmetic and geometric means for triples of numbers and then discuss how they compare.

Visit our website:
www.ContemporaryCollegeAlgebra.org

[7] Notices

1. Discussions of the Contemporary College Algebra program will be held this Fall at the following conferences:

   a. "Mathematics in the Sun," St. Petersburg College, Tarpon Springs, FL, Sept. 26-27, 2003, sponsored by the Florida Two Year College Assoc. Norma Agras, Paul Dirks, Don Small will make presentations.

   b. ICTCM, Chicago, IL, Oct. 30 - Nov. 2, 2003. Don Small is one of the leaders of the preconference meeting that is devoted to refocusing college algebra.

   c. AMATYC, Salt Lake City, UT, Nov. 13 - 16, 2003. Russ Lundgren, Grace Wood, Bob Johnke, Diana Hooker, Don Small will make presentations.

2. Fall Retreat for instructors interested in teaching Contemporary College Algebra will be held at Cy-Fair College in northwestern Houston, TX, October 9-11, 2003. Those interested in attending please contact Laurette Foster (Laurette_Foster@pvamu.edu) or Don Small (don-small@usma.edu). The program will begin with an informal supper Thursday night and conclude with lunch on Saturday. A grant from the Brown Foundation provides financial support for travel, room, and board for the participants.

3. Deadline for contributions to the October Newsletter is Monday, October 6, 2003. Send opinion articles, suggestions for writing assignments, small group in-class activities, small group out-of-class projects, Queries, announcements, etc. to Don Small (don-small@usma.edu).

4. To subscribe to this Newsletter, write to Don Small, Dept. of Mathematics, U.S. Military Academy, West Point, NY 10996 or contact him via e-mail at don-small@usma.edu.