# *Vision - Potential*

*Vision Within Every Instructor – Potential Within Every Student*

Newsletter of the HBCU College Algebra Reform Consortium*
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## [1] Four Millennia of Completing the Square

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Completing the square is a technique for solving quadratic equations that is nearly 4000 years old. That long ago, Mesopotamian scribes pressed the method for posterity into the clay tablets used to record information in ancient Babylonia. Later, during the centuries when Hellenistic Greek culture flourished, Euclid included completing the square in the *The Elements* (c. 300 BC). Arabic algebraists used the method, as will be seen in reviewing the work of Al-Khwarizmi (c. 780-850). In translating and extending Arab works, Renaissance Europeans, to include Cardano (1501-1576), included completing the square as a general method to solve quadratics. It was not until Descartes in the 17th century, however, that the method was finally released from its geometric roots.

Babylonian mathematicians, like their Greek and Arab successors, associated number with length. The origin of geometry, after all, was the measurement of land areas (geo the Greek for earth and metron meaning measure). The multiplication of one length by another was done for applications such as computing the area of land to be planted with barley or calculating the area of an enemy army's encampment for the purpose of estimating the size of the opposing force. There were sufficiently many of these applications that the scribes developed and recorded on tablets specific procedures for solving the resulting equations. One such tablet is BM 13901, which seeks the length of the side given that the sum of the area of a square and 4/3 of the side is 11/12. In modern notation, the equation to be solved is $x^2 + (4/s)x = 11/12$. The solution technique is described in geometric terms, and the method they discovered was completing the square, preserved in solidified clay to this day

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after almost four thousand years.

The ancient Greeks were familiar with this technique developed by Mesopotamian scribes. Greek mathematicians, however, went beyond the Babylonian recipe approach to solving particular problems. They developed general theorems, and provided proofs for those results. Euclid's *Elements*, mostly known for geometric content but which also contains significant algebra and number theory, was compiled around 2300 years ago. The *Elements* contains the method of completing the square in book II, proposition 6:

> If a straight line be bisected and produced to any point, the rectangle contained by the whole line thus produced and the part of it produced, together with the square on half the line bisected, is equal to the square on the straight line which is made up of the half and the part produced.

An associated figure is sketched below:

The relationship between this theorem and the algebraic process of completing the square may be seen in the construction of the square with side $cd$ from the given line segment $ab$. The segment $ab$ is bisected, just as the coefficient $p$ of $x$ is halved in the process of completing the square with the quadratic $x^2 + px = c$. The square with sides of length $cb$ is added to the original problem to complete the square with sides of length $cd$. To convince yourself that this is the method of completing the square, read the extract from *The Elements* and look at the diagram, noting that the length $bd$, the "part of it produced" by extending the original segment $ab$ "to any point," represents the unknown $x$ in the quadratic.

The wording from *The Elements* seems awkward to us, but remember that mathematics was done differently 2000 years ago. A key idea in the evolution of the method demonstrated by Euclid is that he was not solving a specific problem, but instead proving a general mathematical idea. A parallel approach to solving similar problems can be found in the mathematical writings of Arabs who learned from and continued the mathematics of the Hellenic culture.

Al-Khwarizmi was an Arab mathematician who demonstrated the method of completing the square in solving a quadratic, $x^2 + 10x = 39$, in our notation. Al-Khwarizmi did not have our concise way of writing an equation, but had to resort to long verbal explanations and diagrams to express the relationship. The diagram below depicts how Al-Khwarizmi described the process of completing the square.

The figure in the left of the diagram represents the left side of the equation $x^2 + 10x = 39$. The left figure is "completed" by adding the square of area 25, which must be done to the right side of the equation as well to balance the equation. Solving the equation follows by taking the square root of each side.

The title of Al-Khwarizmi's work was Kitab al-jabr waal-muquabala, and it is from the second word in the title that the word algebra is derived. Arab algebraists developed a wide variety of procedures for solving equations. We often use the word *algorithm* to describe procedural methods for solving problems. In fact, mistranslation of Al-Khwarizmi's name is the source of the word algorithm (drop the last i in the name and say the name and then the word).

Leonardo of Pisa, also known as Fibonacci, translated and included Arab mathematics in his work *Liber Abaci* (1202). Girolamo Cardano's *Ars Magna* (1545) was influenced by Fibonacci's compilation, and contains the following problem (chapter 5, problem 2):

> There were two leaders each of whom divided 48 *aurei* among his soldiers. One of these had two more soldiers than the other. The one who had two soldiers fewer had 4 *aurei* more [than the other] for each soldier. What is to be found is how many soldiers each had.

Cardano solves the problem, which can be converted to the equation $x^2 + 2x = 24$ in our notation, by completing the square [Hint: Let $x$ be the smaller number of soldiers].

Our story ends with Cardano. Today, we do not associate solving quadratics with geometric figures (even though the word quadratic is from the Latin *quadratum*, a four-sided figure). Descartes was credited with being the first to label line segments by letters representing the length of the segments, and then multiplying two line segments to obtain a third line segment with length corresponding to the product. This new approach gave Descartes the ability to use algebraic equations to describe a wide variety of curves. For that reason, we associate quadratics with parabolas, not with the squares and rectangles used by those who first conceived and later used the method of completing the squares for the first 3500 years of its application.

### [2] Inquiry: President's Salary Increase - Merit or Cost of Living Increase?

In 1776 when George Washington was elected President, his salary was $25,000. The next President's salary starting in January 2001 will be $400,000. Does this salary increase over the past 224 years represent a merit increase or is it just a Cost-of-Living?

Compute the annual average rate (percent) of increase in the President's salary over the past 224 years. Talk with a professor in Business or Economics or your School's Treasurer or a local Banker about the average rate of inflation over the past 224 years.

Write a paper or make a presentation to your class explaining

a. The meaning of inflation.

b. How to determine the annual average rate of increase in the President's salary.

c. Your conclusion as to whether the increase in the President's salary represents merit increase or a Cost-of-Living increase.

### [3] The Plotting Spider

Consider a room is 30 feet long (east to west), 12 feet wide (north to south), and 12 feet high. A spider on the centerline of the west wall of the room, one foot above the floor, sees a fly asleep on the centerline of the east wall, one foot below the ceiling. The spider wants to get to the fly as quickly as possible. Which is the shortest path for the spider to take to get to the fly and what is the length of this path? (The spider must always be in touch with the floor, ceiling, or a wall - no flying spiders.)

### [4] It Might Be Simpler To Walk

Elmer read an ad in *Sports Illustrated* offering a "real deal." It read

> A 2000 Dodge Intrepid with 0.0% financing for 60 months or a $2,000 Cash Allowance.

In fine print, the offer required 10% down payment and 4.9% APR financing with the cash allowance. Here are a few questions for Elmer to consider, assuming the ticket price is $30,000 and the balance after the down payment is financed:

a. How much is the down payment?

b. How much is the monthly payment under the 0.0% financing plan?

c. How much is the monthly payment under the 4.9% APR financing plan? Hint: See Problem #2 in Section 3.2 of the *Contemporary College Algebra* text.

d. How much interest is paid over 60 months under the 4.9% APR financing plan?

e. Which plan benefits Elmer most when inflation is considered? Why?

f. What APR interest rate would Elmer have to receive if he were to earn the total (car) interest pay-out by investing the $2,000 cash allowance for 60 months?

## [5] Query

A sign over a knick-knack table at a fair proclaimed

**10% Discount on all Sales $10 or More**

Margaret, who had $10 to spend, became intrigued when she saw the sign and wondered how much she could buy (total price) and still pay only $10. What is the answer to her bewilderment?

## [6] College Algebra Reform at the 2001 National Mathematics Meetings

The National Mathematics Meetings that will be held in New Orleans, January 10-13, 2001, will offer several programs directed at College Algebra Reform. The activities include:

a. Minicourse #12: "Contemporary College Algebra, a Reform Program." Organized by Laurette Foster (Prairie View A & M Univ.), Dorothy Hunter (Huston-Tillotson College), Don Small (US Military Academy) Wednesday and Friday mornings 8-10:00 a.m.

b. Contributed Paper Session: "Redefining What a Modern College Algebra Experience Means." Wednesday and Thursday mornings.

c. Panel: "Redefining College Algebra Courses" organized by Shelly Gordon. Panelists include Alex Fluellen (Clark Atlanta Univ.), Don Small (US Military Academy). Wednesday 2:15-3:45 p.m.

d College Algebra Reform Poster Session organized by Dorothy Hunter and General Marshall (Huston-Tillotson College), Don Small (US Military Academy). Friday 9-11:00 a.m.

e. "Open Discussion on Reforming College Algebra" Organized by the HBCU Consortium for Reforming College Algebra. Panalists: Della Bell (Texas Southern Univ.) and Sarah Bush (Wiley College). Saturday 2:45-4:15 p.m.

f. Annual meeting of Local Coordinators and Instructors of Contemporary College Algebra. Friday 2:30-5:00 p.m.

There are a number of other sessions in which the subjects being discussed have a bearing on reforming College Algebra such as the Panel Discussion on "How to Facilitate Change."

## [7] Notices

1. The next issue of the *Vision - Potential* Newsletter will be published in January 2001. The Deadline for contributions to the January Newsletter is Monday, January 8, 2001.

2. To subscribe to this Newsletter or to submit articles write to Dr. Della Bell, Chair, Dept. of Mathematics, Texas Southern University, 3100 Cleburne St., Houston, TX 77004.